COMPUTATIONAL–EXPERIMENTAL METHOD FOR SOLVING TEMPERATURE PROBLEMS WITH COORDINATE- AND TIME-DEPENDENT BOUNDARY CONDITIONS

Yu. I. Aryutkin, V. F. Kuryakin, and Yu. K. Semenov

UDC 536.2

A method for finding the temperature fields and for identifying the conditions of heat exchange for constructions with coordinate- and time-dependent boundary conditions is presented. Inverse problems are reduced to systems of convolution-type integral equations.

A problem arising in the development of complicated constructions is the problem of determining and setting admissible operating conditions of the structures for diverse thermal perturbations.

Calculations and experiments are used to find the temperature fields and stress–strain states and the limiting values of the criterional parameters of parts of the constructions. As a rule, the calculations of complicated constructions must be calibrated with the help of special experiments, only a limited number of direct experiments, and in a simplified form that does not encompass all the diverse thermal perturbations, can be performed.

For this reason, it is useful to reduce the experimental data directly to real operating conditions and to formulate requirements on the form and volume of the minimum necessary experiments. This can be done reliably and economically by experimental–theoretical methods.

The experimental–theoretical method and the TEZIS program which implements it on a computer [1, 2] are intended for determining the temperature fields of constructions and for identifying the boundary conditions of heat transfer, including in the case of time-dependent temperature and heat-emission coefficients of the medium. They make it possible to compare two experiments using an integral equation

$$g_1 * v_2 = g_2 * v_1 \tag{1}$$

or the more general equation

$$Ag_1 * v_2 = g_2 * v_1 + (hg_2) * g_1, \tag{2}$$

where

$$g * v = \int_0^\tau g(\tau - \omega)\, v(\omega)\, d\omega = v * g;$$

$g_1 = g_1(k, \tau)$ is the response (temperature field) of the construction to the perturbation $v_1 = v_1(\tau)$ (change in time of the temperature of the surface, the heat flux, or temperature of the medium with a constant coefficient of heat emission α); $g_2 = g_2(k, \tau)$ is the response to a perturbation $v_2 = v_2(\tau)$; and, A and H are functions which are determined by the heat-emission coefficients.

In Eq. (1) the unknown can be any of the four functions, and because the convolution is commutative the responses and perturbations (effects and causes) can be interchanged. This problem is interpreted as a problem with one input.

The method requires that a number of conditions by satisfied. The heat-conduction problem is assumed to be homogeneous and internally linear. The thermophysical properties of the structural parts (which need not be known) must be the same in both experiments (within the limits of thermal similarity), and the initial temperature must be reduced to

Translated from Inzhenerno-fizicheskii Zhurnal, Vol. 61, No. 3, pp. 479-484, September, 1991. Original article submitted August 21, 1990.

zero. It is important that in the boundary conditions the spatial distribution of the external perturbations, for example, the distribution of the temperature profiles on the surface, must exhibit similarity. This is usually achieved by working with the average surface temperature in zones determining the temperature of the internal elements of interest, but in the general case a different treatment is allowed.

In the experimental–theoretical method of determining the temperature fields of constructions under coordinate- and time-dependent boundary conditions the solution of the problem can be written in the form of a system of integral equations:

$$1*g_l = \sum_{r=1}^{L} g_r^l * v_{lr}, \; l = 1, 2, \ldots, L+1, \tag{3}$$

where L is the number of constant perturbations along the coordinate, for example, the number of isothermal zones on the surface; $g_r^I = g_r^I(k, \tau)$ is the response (temperature field) of the structure to a unit perturbation $v_r = 1$, which is constant in time and as a function of the coordinate, on the part r of the surface and is equal to zero elsewhere on the surface; $v_{\ell r} = v_{\ell r}(\tau)$ is the perturbation in the ℓ-th experiment and is constant on the part r of the surface and is time dependent; $g_l = g_l(k, \tau)$ is the response of the construction in the l-th experiment to the time-dependent total perturbation $\sum_{r=1}^{L} v_{\ell r}$ which is different on different parts r of the surface.

The number of isothermal zones is chosen in the usual manner, starting from the desired approximation accuracy. The locations where the zones meet the arithmetic-mean values of the perturbations are used. In Eq. (3) any component can be found. The number of equations obviously must not be less than the number of unknowns, and thus the minimum number of required experiments is equal to the number of isothermal zones, if there is no symmetry (see below). In addition, the experiments, in contrast to the experiments described by Eq. (1), must not be similar to one another along the perturbation profiles on the surface of the structure. Otherwise the same conditions as in the first method must be satisfied.

The difficulty of using Eqs. (3) in practice lies in the fact that special experiments must be performed in order to obtain responses to a perturbation that is constant in time and as a function of the coordinate on part of the surface and is equal to zero elsewhere on the surface, i.e., so-called transfer functions, and in order to find these functions by solving inverse problems on the basis of data on g_l and $v_{\ell r}$ a large volume of calculations, in which errors accumulate, must be performed.

If the number of coordinates of the interior points of the construction is equal to the number of zones L of the perturbations and the number of experiments K = L, the system (3) can be approximated by the finite-dimensional block-matrix approximation

$$I*G = G^I*V, \tag{4}$$

and the interesting (L + 1) case can be written, correspondingly, as

$$1*g = G^I*v, \tag{5}$$

where $G^I = [g_{kri}^I]$, $G = [g_{kri}]$ are the matrices of the responses to a unit matrix I of the perturbations and the matrix of perturbations $V = [v_{ri}]$, respectively, while $g = \{g_{ki}\}$ is the vector of responses to the vector of perturbations $v = \{v_{ri}\}$.

The convolution of matrices (and matrices with vectors) is performed by the same rules as multiplication of matrices and vectors, except that instead of multiplication a convolution is performed. Thus the expression (5) for the coordinate k at the time i can be represented, for example, as

$$1*g_{ki} = \sum_{r=1}^{L} \sum_{j=1}^{i} g_{kr(i-j-1)}^I v_{rj}. \tag{6}$$

In order to find from Eq. (4) the matrices of the transfer functions G for Eq. (5) it is necessary to solve inverse problems for L systems of equations with L unknowns in each system. If Eq. (5) is then solved for v, then another inverse problem for a system with L unknowns is added. Combining Eqs. (4) and (5) makes it unnecessary to find the matrices of the transfer functions and leads to the equation

$$G^{-1} * g = V^{-1} * v = u, \tag{7}$$

where G^{-1} and V^{-1} are inverse matrices to the matrices G and V, and $u = \{u_{ri}\}$ is the two-dimensional conversion vector.

Two systems of equations are obtained from Eq. (7):

$$G * u = 1 * g, \tag{8}$$

$$V * u = 1 * v \tag{9}$$

and the problem reduces to the inverse problem of finding u from these systems and the direct problem of determining the responses g or perturbations v from the system (8) or (9), respectively, differentiated with respect to time. To find v it is better to use the temperatures of the zones on the surface of the construction and not at interior points.

It is evident that there is an analogy between the system (7) and Eq. (1), but these problems can now be interpreted as problems with several (L) inputs. In the equations the perturbations are presented in the same form for each zone of the construction: boundary conditions of the first kind (temperatures G^{surf} of zones on the surface), second kind (heat fluxes Q flowing to zones on the surface), or third kind (temperatures G^{med} of the external medium at the zones of the surface) with heat-emission coefficients $\alpha_{r\ell}$, $\ell = 1, 2, \ldots, L$, which are constant in time and are identical in all experiments for each zone.

In order to convert the temperature fields of the construction from one set of heat-transfer conditions with respect to the temperatures of the medium and the heat-emission coefficients in L experiments to different fields and coefficients g^{med} and α_l = const, $l = 1, 2, \ldots, L$, it is necessary to employ in the (L + 1) experiment (case) in the equations the relation between the boundary conditions of the second and third kinds. In the case when the heat flux is related linearly with the difference of the temperatures of the medium and the surface (Newton's hypothesis)

$$[q_{rli}] = [\alpha_{rli}(g^{med}_{rli} - g^{surf}_{rli})] \tag{10}$$

we obtain from Eq. (7), using Eq. (10),

$$[DQ + G^{surf}] * u = 1 * g^{med}, \tag{11}$$

where $Q = [q_{r\ell i}]$, $G^{surf} = [g_{r\ell i}{}^{surf}]$ are matrices of the heat fluxes and surface temperatures in the experiments; $g^{med} = \{g_{ri}{}^{med}\}$ is the vector of temperatures of the medium in the case of interest; D is a diagonal matrix, consisting of quantities that are the inverses of the heat-emission coefficients in the case of interest.

The temperatures of the surface and at internal points of the construction are calculated as before by solving Eq. (8) after u has been found.

The case of perturbations that are constant in time but are different on different sections of the surface of the construction is a particular case and is obtained from Eqs. (8), (9), and (11) by replacing the convolution operations by multiplication (a system of linear algebraic equations is solved). The variant in which the perturbations are time dependent and spatially similar in two experiments also follows from these equations and is described by Eq. (1).

The problems are simplified and the volume of calculations is reduced if the construction has a center, axis, or plane of symmetry. In this case its surface is partitioned into equal areas, arranged symmetrically with respect to its zone center (axis). An appropriate choice of points at which the temperature is measured, also arranged symmetrically, within the construction makes it possible to reduce the number of experiments, right down to one experiment, but with coordinate-dependent (and time-dependent, but better constant in time) perturbation. The missing elements in the matrices are filled in according to the principle of cyclic substitution, based on the symmetry of the results, with successive imaginary rotations of the construction or perturbations (equal to the number of zones); this procedure is equivalent to new experiments. In the process, in separate neighboring zones the perturbations can be equal to one another. It is only necessary that the convolution determinant of the system be different from zero.

The inverse problems (5), (8), (9), and (11) are described identically by the system

$$A * u = f, \tag{12}$$

where A and f are a three-dimensional matrix and the two-dimensional vector of initial data and u is the two-dimensional solution vector.

The system (12) can be conveniently solved in terms of the convolution determinants, i.e., determinants which are calculated by replacing multiplication by convolution. After these determinants are found the inverse problems are solved:

$$\Delta * u_r = \Delta_r, \tag{13}$$

where Δ and Δ_r are the convolution determinant of the matrix A and the determinant obtained from Δ by replacing the elements in the r-th column by elements of the vector f; $u_r = u_r(\tau)$ is the one-dimensional solution vector for the coordinate r.

The system (13) can be combined with (8). Then

$$\Delta * g = \sum_{r=1}^{L} g_r * \Delta_r, \tag{14}$$

which is convenient computationally: the right-hand side is found and the inverse problem of determining g is solved.

The method of solving inverse problems with precalculation of the convolution determinants has certain limitations. If the time expended on the operation of one convolution is proportional to the square of the number of time points and is not too long (seconds), then in calculating only one determinant (using Cramers rule) the number of operations increases approximately as the factorial of the dimension of the determinant. It is obvious that for more than four or five zones (and when the structure has no symmetry), a different computational strategy must be employed. In this case, iteration methods, for example, such as Gold's method and the method of directed divergence, can be used to obtain an approximate solution of the system (12), as in the case of the systems (1) and (2). In these methods the n + 1 approximation is found as

$$u^{n+1} = u^n \frac{A^T * f}{A^T * (A * u)}, \tag{15}$$

where A^T is transpose of matrix A.

The advantage of the formulas (12)-(15) lies in the compactness and uniformity of the calculations. This has been implemented in the REMIS program.

The program processes the experimentally obtained starting data, which are given at unequally spaced moments in time, by normalizing, smoothing when necessary, and interpolating the data so as to achieve a uniform time grid. Cubic splines are used for smoothing and exponential slines are used for interpolation.

The possible symmetry of the structure is taken into account. Then it is sufficient to introduce starting data for one experiment, starting from the principle of cylic substitution. The computing time is proportional to the squared number of time points, the number of zones, and the number of iterations. For a SM1420 computer with 100 iterations and 12 zones with 25 time points the computing time is equal to approximately 50 min.

The calculation of the determinants reduces to integration procedures and if the computational step is sufficiently small, it introduces virtually no errors (the step is chosen on the basis of the desired accuracy of integration by the trapezoidal method). The iteration methods for solving the systems terminate when a prescribed degree of discrepancy is achieved. Numerical modeling [1, 3] of the accuracy of the solutions obtained for the inverse problems by the methods employed showed that the accuracy is not lower than the accuracy of the initial data. The program contains test samples, in which simple analytic functions (of the exponential type) are employed as the initial data.

The application of the two methods, described above, for determining the temperature fields of constructions and a comparison of the results obtained using them can be clearly illustrated for a model example. Let the structure be symmetric (infinite cylinder or a sphere) and let its surface be partitioned by a plane into two (left- and right-hand)

zones of equal area. The temperatures of the zones in the first experiment vary in time as $v_{11} = [1 - \exp(-\tau)]$ and $v_{21} = [1 - \exp(-2\tau)]$ (the maximum difference is equal to 25% and appears at $\tau = 0.7$). Inside the structure at points of interest k_1 and k_2, arranged symmetrically at equal distances and under the same angles relative to the center, the temperatures are $g_{11} = [1 - \exp(-0.3\tau)]$, $g_{21} = [1 - \exp(-0.4\tau)]$ (the largest difference is equal to 10% and occurs at $\tau = 2.9$), and the temperature at the center, the point k_0, is equal to $g_{01} = [1 - \exp(-0.1\tau)]$. Since there is symmetry, a second experiment can be imagined with the conditions $v_{12} = v_{21}$; $v_{22} = v_{11}$ and therefore $g_{12} = g_{21}$, $g_{22} = g_{11}$, and $g_{02} = g_{01}$. The limiting case in which the change in the temperature is the same on the entire surface of the structure and, in particular, the case close to a jump-like change $v_1 = v_2 = [1 - \exp(-p\tau)]$ as $p \to \infty$, is of interest. Then from Eqs. (9) and (14) we obtain

$$g_k * \Delta = (g_{k1} + g_{k2}) * \Delta_1, \tag{16}$$

$$g_k * (v_{11} + v_{12}) = (g_{k1} + g_{k2}) * v_1 \tag{17}$$

with $k = k_1$, $k = k_2$, $k = k_0$, respectively, i.e., the temperatures at the points of interest can also be found from Eq. (1), and in so doing it is convenient to employ the arithmetic-mean values $(v_{11} + v_{12})/2$ and $(g_{k1} + g_{k2})/2$.

For the functions used in the example to describe the changes in the temperatures of the structure in the experiments 1 and 2, analytic solutions determining the temperature of the construction are obtained from Eqs. (1) and (4) with a practically jump-like change in the temperature of the left-hand, right-hand, and both isothermal zones of the surface. Comparison shows that the discrepancies in the solutions do not exceed 5% of the average values. If, however, the zones v_2 and v_1 of the surface are used to determine the temperature far from the points k_1 and k_2, then the discrepancies increase to 10%, but from the physical standpoint this is not justified.

NOTATION

Here τ is the time; $k = k(x, y, z)$ is the coordinate; r, l, and i are the instantaneous number of isothermal coordinate zones of the thermal perturbation, the experiment, and the time point; R, L, and N are the total number of coordinates of the zones, experiments, and time points; $g_k{}^l$ is the temperature in the l-th experiment at the coordinate k; $g_{kr}{}^I$ is the temperature at the coordinate k with a unit jump-like perturbation in the zone r and zero perturbation in other zones; g(med/re), g(surf/re), $v_r{}^l$, $q_r{}^l$, $\alpha_r{}^l$ are the temperature of the medium, the temperature of the surface, the perturbation, the heat flux, and the heat-emission coefficient in the r-th zone and the l-th experiment; G, V, and Q are three-dimensional matrices of the temperatures, perturbations, and heat fluxes; Δ is the convolution determinant; *, I, T, and -1 designate convolution, unit perturbation, transposition, and inversion.

LITERATURE CITED

1. Yu. I. Aryutkin, V. F. Kuryakin, and Yu. K. Semenov, in: Methods and Programs for Numerical Solution of Problems in Mathematical Physics [in Russian], No. 3, Moscow (1985), pp. 52-55.
2. Yu. I. Aryutkin, V. F. Kuryakin, and Yu. K. Semenov, Inzh.-fiz. Zh., 57, No. 4, 685 (1989).
3. O. M. Alifanov, Identification of Heat-Transfer Processes in Aircraft [in Russian], Moscow (1979).